Understanding Pure Mathematics

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Exercise 20B

(1) Let $I = \int x(x+1)^4 \, dx$

Let $u = x$, $\therefore du = 1\,dx$

$dv = (x+1)^4 dx$, $\therefore v = \frac{1}{5}(x+1)^5$

So $I = \frac{x}{5}\cdot(x+1)^5 - \int 1\cdot\frac{1}{5}(x+1)^5 \, dx$

$= \frac{x}{5}(x+1)^5 - \frac{1}{30}(x+1)^6 + c$

(2) Let $I = \int x\cdot\sqrt{x-3} \, dx$

Let $u = x$, $\therefore du = 1\,dx$

$dv = \sqrt{x-3}\,dx = (x-3)^{1/2} dx$, $\therefore v = \frac{2}{3}(x-3)^{3/2}$

So $I = \frac{2}{3}x\cdot(x-3)^{3/2} - \int 1\cdot\frac{2}{3}(x-3)^{3/2} dx$

$= \frac{2}{3}x\cdot(x-3)^{3/2} - \frac{4}{15}(x-3)^{5/2} + c$

(3) let $I = \int \frac{x}{\sqrt{x+3}} \, dx = \int x \cdot (x+3)^{-\frac{1}{2}} \, dx$

let $u = x$, $\therefore du = 1 \, dx$

$dv = (x+3)^{-\frac{1}{2}} dx$, $\therefore v = 2(x+3)^{\frac{1}{2}}$

so $I = 2x.(x+3)^{\frac{1}{2}} - \int 1.2.(x+3)^{\frac{1}{2}} \, dx$

$= 2x(x+3)^{\frac{1}{2}} - \frac{4}{3}(x+3)^{\frac{3}{2}} + c$

(4) let $I = \int x \cdot \sin x \, dx$

let $u = x$, $\therefore du = 1 \, dx$

$dv = \sin x \, dx$, $\therefore v = -\cos x \, dx$

so $I = -x . \cos x - \int 1 . (-\cos x) \, dx$

$= -x . \cos x + \sin x + c$

(5) let $I = \int x^2 \sin x \, dx$

let $u_1 = x^2$, $\therefore du_1 = 2x \, dx$

$dv_1 = \sin x \, dx$, $\therefore v_1 = -\cos x$

so $I = -x^2 . \cos x - \int 2x(-\cos x) \, dx$

$= -x^2 \cos x + 2 \int x . \cos x \, dx$

Now let $u_2 = x$, $\therefore du_2 = 1 \cdot dx$

$dv_2 = \cos x \, dx$, $\therefore v_2 = \sin x$

So $I = -x^2 \cdot \cos x + 2\left[x \sin x - \int 1 \cdot \sin x \, dx\right]$

$= -x^2 \cdot \cos x + 2x \sin x + 2\cos x + c$

(6) let $I = \int 2x \sin(3x-1)\, dx$

let $u = 2x$, $\therefore du = 2 . dx$

$dv = \sin(3x-1)\, dx$, $\therefore v = -\frac{1}{3}\cos(3x-1)$ (by Substitution)

So $I = -\frac{2}{3}\cos(3x-1) - \int 2 \cdot \left(-\frac{1}{3}\cos(3x-1)\right) dx$

$= -\frac{2}{3}\cos(3x-1) + \frac{2}{9}\sin(3x-1) + c$

(7) let $I = \int x^2 . \ln x \, dx$

let $u = \ln x$, $\therefore du = \frac{1}{x}\, dx$

$dv = x^2 \, dx$, $\therefore v = \frac{x^3}{3}$

So $I = \frac{x^3}{3} . \ln x - \int \frac{x^3}{3} \cdot \frac{1}{x}\, dx$

$= \frac{x^3}{3} . \ln x - \frac{1}{9}x^3 + c$

(8) let $I = \int (x+1) \ln x \, dx$

let $u = \ln x$, $\therefore du = \frac{1}{x} dx$

$dv = (x+1) dx$, $\therefore v = \frac{x^2}{2} + x$

So $I = \left(\frac{x^2}{2} + x\right) \ln x - \int \frac{1}{x} \left(\frac{x^2}{2} + x \right) dx$

$= \left(\frac{x^2}{2} + x\right) \ln x - \frac{x^2}{4} - x + c$

(9) let $I = \int \frac{1}{x^3} . \ln x \, dx$

let $u = \ln x$, $\therefore du = \frac{1}{x} dx$

$dv = \frac{1}{x^3} = x^{-3}$, $\therefore v = -\frac{1}{2} x^{-2}$

So $I = -\frac{1}{2} \cdot \frac{1}{x^2} \cdot \ln x - \int \left(-\frac{1}{2} \frac{1}{x^2}\right) \cdot \frac{1}{x} dx$

$= -\frac{1}{2x^2} \cdot \ln x - \frac{1}{4} \cdot \frac{1}{x^2} + c$

(10) let $I = \int e^x \sin x \, dx$

let $u_1 = e^x$, $\therefore du_1 = e^x dx$

$dv_1 = \sin x \, dx$, $\therefore v_1 = -\cos x dx$

So $I = -e^x \cos x - \int e^x \cdot (-\cos x) dx$

let $u_2 = e^x$, $\therefore du_2 = e^x dx$

$dv_2 = \cos x \, dx$, $\therefore v_2 = \sin x$

So $I = -e^x \cos x + \left[e^x . \sin x - \int e^x . \sin x \, dx \right]$

$= e^x \sin x - e^x \cos x - I$

So $2I = e^x . \sin x - e^x \cos x \Rightarrow I = \frac{1}{2}(e^x . \sin x - e^x \cos x)$

(11) let $I = \int x . e^x dx$

let $u = x$, $\therefore du = 1 . dx$

$dv = e^x dx$, $\therefore v = e^x$

So $I = x e^x - \int 1 . e^x dx = x e^x - e^x + c$

(12) let $I = \int x^2 . e^{3x} dx$

let $u_1 = x^2$, $\therefore du_1 = 2x \, dx$

$dv_1 = e^{3x}$, $\therefore v_1 = \frac{1}{3} e^{3x}$

So $I = \frac{1}{3} x^2 . e^{3x} - \frac{2}{3} \int x . e^{3x} dx$

Now let $u_2 = x$, $\therefore du_2 = 1 . dx$

$dv_2 = e^{3x}$ ... e^{3x}

So $I = \frac{1}{3}x^2 \cdot e^{3x} - \frac{2}{3}\left[\frac{1}{3}x \cdot e^{3x} - \int 1 \cdot \frac{1}{3}e^{3x}\,dx\right]$

$= \frac{1}{3}x^2 \cdot e^{3x} - \frac{2}{9}x e^{3x} + \frac{2}{27}e^{3x} + c$

(13) let $I = \int e^x \cdot \cos 2x\,dx$

Let $u_1 = \cos 2x$, $\therefore du_1 = -2\sin 2x\,dx$

$dv_1 = e^x\,dx$, $\therefore v_1 = e^x$

So $I = e^x \cos 2x - \int e^x \cdot (-2\sin 2x)\,dx$

Now let $u_2 = \sin 2x$, $\therefore du_2 = 2\cos 2x\,dx$

$dv_2 = e^x\,dx$, $\therefore v_2 = e^x$

So $I = e^x \cdot \cos 2x + 2\left[e^x \sin 2x - 2\int e^x \cdot \cos 2x\,dx\right]$

$= e^x \cdot \cos 2x + 2e^x \cdot \sin 2x - 4I + c$

So $5I = e^x \cdot \cos 2x + 2e^x \sin 2x + c$

$\Rightarrow I = \frac{1}{5}e^x(\cos 2x + 2\sin 2x) + k$, where $k = c/5$

(14) let $I = \int \ln(2x+1)\,dx = \int 1.\ln(2x+1)\,dx$

let $u = \ln(2x+1)$, $\therefore du = \frac{2}{2x+1}\,dx$.

$dv = 1\,dx$, $\therefore v = x$

So $I = x.\ln(2x+1) - 2\int \frac{x}{2x+1}\,dx$

$= x.\ln(2x+1) - 2\int \frac{1}{2} - \frac{1/2}{2x+1}\,dx$ by long division

$= x.\ln(2x+1) - 2\left[\frac{1}{2}x - \frac{1}{4}\ln(2x+1)\right] + c$

$= x.\ln(2x+1) - x + \frac{1}{2}\ln(2x+1) + c$

$= \frac{1}{2}(2x+1)\ln(2x+1) - x + c$

(15) let $I = \int \tan^{-1}x\,dx = \int 1.\tan^{-1}x\,dx$

let $u = \tan^{-1}x$, $\therefore du = \frac{1}{1+x^2}\,dx$

$dv = 1\,dx$, $\therefore v = x$

So $I = x.\tan^{-1}x - \int \frac{x}{1+x^2}\,dx$

$= x.\tan^{-1}x - \frac{1}{2}\ln(1+x^2) + c$

(16) let $I = \int \sin^{-1} x \, dx = \int 1 \cdot \sin^{-1} x \, dx$

let $u_1 = \sin^{-1} x$, $\therefore du_1 = \frac{1}{\sqrt{1-x^2}} dx$

$dv_1 = 1$, $\therefore v_1 = x$

So $I = x \cdot \sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}} dx$

Now let $u_2 = 1 - x^2 \Rightarrow du_2 = -2x \, dx$

$\therefore -\frac{1}{2} du_2 = x \, dx$

$\therefore \int \frac{x}{\sqrt{1-x^2}} dx = -\frac{1}{2} \int \frac{1}{\sqrt{u_2}} \cdot du_2$

$= -u_2^{1/2} = -(1-x^2)^{1/2}$

So $I = x \cdot \sin^{-1} x + (1-x^2)^{1/2} + c$

(17) let $I = \int 2x \cdot \ln(x+1) \, dx$

let $u = \ln(x+1)$, $\therefore du = \frac{1}{x+1} dx$

$dv = 2x$, $\therefore v = x^2$

So $I = x^2 \cdot \ln(x+1) - \int \frac{x^2}{x+1} dx$

$= x^2 \cdot \ln(x+1) - \int x - 1 + \frac{1}{x+1} dx$ by long division

So $I = x^2 \cdot \ln(x+1) - \left[x^2 - x + \ln(x+1)\right] + c$

$= (1+x^2)\ln(x+1) - x^2 + x + c$

(18) let $I = \int x \cdot \ln(x-2)\, dx$

let $u = \ln(x-2)$, $\therefore du = \frac{1}{x-2}$

$dv = x$, $\therefore v = \frac{x^2}{2}$

So $I = \frac{x^2}{2} \cdot \ln(x-2) - \frac{1}{2}\int \frac{x^2}{x-2}\, dx$

$= \frac{x^2}{2} \cdot \ln(x-2) - \frac{1}{2}\int x+2 + \frac{4}{x-2}\, dx$ by long division

$= \frac{x^2}{2} \cdot \ln(x-2) - \frac{1}{2}\left[\frac{x^2}{2} + 2x + 4\ln(x-2)\right] + c$

$= \frac{1}{2}(x^2-4)\ln(x-2) - x - \frac{x^2}{4} + c$

(19) let $I = \int_0^1 x(x-1)^5\, dx$

Let $u = x$, $\therefore du/dx = 1$

and $\frac{dv}{dx} = (x-1)^5$ $v = \frac{1}{6}(x-1)^6$

$\therefore I = x \frac{1}{6} \cdot (x-1)\Big|_0^1 - \frac{1}{6}\int_0^1 (x-1)^6\, dx$

$= \left[\frac{x}{6} \cdot (x-1) - \frac{1}{42}(x-1)\right]_0^1 = 0 - \left(-\frac{1}{42}(-1)\right) = \frac{1}{42}$

(20) Let $I = \int_3^6 \frac{x}{\sqrt{x-2}}\,dx = \int_3^6 x\cdot(x-2)^{-\frac{1}{2}}\,dx$

Then let $u = x$, $\therefore \frac{du}{dx} = 1$

and $\frac{dv}{dx} = (x-2)^{-\frac{1}{2}}$, $\therefore v = 2(x-2)^{\frac{1}{2}}$

So $I = 2x(x-2)^{\frac{1}{2}}\Big|_3^6 - \int_3^6 2(x-2)^{\frac{1}{2}}\,dx$

$= \left[2x\cdot(x-2)^{\frac{1}{2}} - \frac{4}{3}(x-2)^{3/2}\right]_3^6$

$= \left[12(4)^{\frac{1}{2}} - \frac{4}{3}(8)\right] - \left[6(1) - \frac{4}{3}(1)\right] = 8\tfrac{2}{3}$

(21) let $I = \int_0^2 x(x-2)^4\,dx$

let $u = x$, $\therefore \frac{du}{dx} = 1$

& $\frac{dv}{dx} = (x-2)^4$, $\therefore v = \frac{1}{5}(x-2)^5$

$\therefore I = \frac{x}{5}\cdot(x-2)^5\Big|_0^2 - \frac{1}{5}\int_0^2 (x-2)^5\,dx$

$= \left[\frac{x}{5}\cdot(x-2)^5 - \frac{1}{30}(x-2)^6\right]_0^2 = 0 - \left(-\frac{1}{30}(-2)\right) = 2\tfrac{2}{15}$

(22) Let $I = \int_0^{\pi/4} x \cdot \cos 2x \, dx$

Now let $u = x$, $\therefore \frac{du}{dx} = 1$

& $\frac{dv}{dx} = \cos 2x$, $\therefore v = \frac{1}{2} \sin 2x$

So $I = \frac{x}{2} \cdot \sin 2x \Big|_0^{\pi/4} - \frac{1}{2}\int_0^{\pi/4} \sin 2x \, dx$

$= \left[\frac{x}{2} \cdot \sin 2x + \frac{1}{4} \cos 2x \right]_0^{\pi/4} = \left[\frac{\pi}{8}(1) + 0 \right] - \left[0 + \frac{1}{4} \right]$

$= \frac{1}{8}(\pi - 2)$

(23) Let $I = \int_2^4 (x-1) \ln(2x) \, dx$

Now, let $u = \ln(2x)$, $\therefore \frac{du}{dx} = \frac{1}{x}$

& $\frac{dv}{dx} = (x-1)$, $\therefore v = \frac{(x-1)^2}{2}$

$\therefore I = \frac{1}{2}(x-1)^2 \cdot \ln(2x) \Big|_2^4 - \frac{1}{2}\int_2^4 \frac{(x-1)^2}{x} \, dx$

$= \frac{1}{2} \cdot (x-1)^2 \cdot \ln(2x) \Big|_2^4 - \frac{1}{2}\int_2^4 \frac{x^2 - 2x + 1}{x} \, dx$

$= \frac{1}{2} \cdot (x-1)^2 \cdot \ln(2x) \Big|_2^4 - \frac{1}{2}\int_2^4 x - 2 + \frac{1}{x} \, dx$

$= \left[\frac{1}{2}(x-1)^2 \ln(2x) - \frac{1}{4}x^2 + x - \frac{1}{2}\ln x \right]_2^4$

(24) Let $I = \int_1^4 \frac{\ln x}{x^2}\, dx$

Now let $u = \ln x$, $\therefore \frac{du}{dx} = \frac{1}{x}$

& $\frac{dv}{dx} = \frac{1}{x^2}$, $\therefore v = -\frac{1}{x}$

So $I = -\frac{\ln x}{x}\Big|_1^4 + \int_1^4 \frac{1}{x^2}\, dx = \left[-\frac{\ln x}{x} - \frac{1}{x}\right]_1^4$

$= \left(-\frac{\ln 4}{4} - \frac{1}{4}\right) - (\ln 1 - 1)$

$= \frac{1}{4}(3 - \ln 4)$

(25) Let $I = \int_0^2 x^3 \cdot e^x\, dx$

Now let $u_1 = x^3$, $\therefore \frac{du_1}{dx} = 3x^2$

& $\frac{dv_1}{dx} = e^x$, $\therefore v_1 = e^x$

$\therefore I = x^3 e^x\Big|_0^2 - 3\int_0^2 x^2 \cdot e^x\, dx$

Now let $u_2 = x^2$, $\therefore \frac{du_2}{dx} = 2x$

& $\frac{dv_2}{dx} = e^x$, $\therefore v_2 = e^x$

$\therefore I = x^3 e^x\Big|_0^2 - 3\left[x^2 e^x\Big|_0^2 - 2\int_0^2 x e^x\, dx\right]$

Now let $u_3 = x$, $\frac{du_3}{dx} = 1$

$\frac{dv_3}{dx} = e^x$, $v_3 = e^x$

So $I = x^3 e^x \Big|_0^2 - 3\left[x^2 e^x \Big|_0^2 - 2\left(xe^x - e^x\right)_0^2\right]$

$= \left[x^3 e^x - 3x^2 e^x + 6xe^x - 6e^x\right]_0^2$

$= (8e^2 - 12e^2 + 12e^2 - 6e^2) - 6 = 2e^2 + 6$